

# WEYL'S LAW IN THE THEORY OF AUTOMORPHIC FORMS

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**ABSTRACT.** For a compact Riemannian manifold, Weyl's law describes the asymptotic behavior of the counting function of the eigenvalues of the associated Laplace operator. In this paper we discuss Weyl's law in the context of automorphic forms. The underlying manifolds are locally symmetric spaces of finite volume. In the non-compact case Weyl's law is closely related to the problem of existence of cusp forms.

## 1. INTRODUCTION

Let  $M$  be a smooth, compact Riemannian manifold of dimension  $n$  with smooth boundary  $\partial M$  (which may be empty). Let

$$\Delta = -\operatorname{div} \circ \operatorname{grad} = d^*d$$

be the Laplace-Beltrami operator associated with the metric  $g$  of  $M$ . We consider the Dirichlet eigenvalue problem

$$(1.1) \quad \Delta\phi = \lambda\phi, \quad \phi|_{\partial M} = 0.$$

As is well known, (1.1) has a discrete set of solutions

$$0 \leq \lambda_0 \leq \lambda_1 \leq \cdots \rightarrow \infty$$

whose only accumulation point is at infinity and each eigenvalue occurs with finite multiplicity. The corresponding eigenfunctions  $\phi_i$  can be chosen such that  $\{\phi_i\}_{i \in \mathbb{N}_0}$  is an orthonormal basis of  $L^2(M)$ . A fundamental problem in analysis on manifolds is to study the distribution of the eigenvalues of  $\Delta$  and their relation to the geometric and topological structure of the underlying manifold. One of the first results in this context is Weyl's law for the asymptotic behavior of the eigenvalue counting function. For  $\lambda \geq 0$  let

$$N(\lambda) = \#\{j: \sqrt{\lambda_j} \leq \lambda\}$$

be the counting function of the eigenvalues of  $\sqrt{\Delta}$ , where eigenvalues are counted with multiplicities. Denote by  $\Gamma(s)$  the Gamma function. Then the Weyl law states

$$(1.2) \quad N(\lambda) = \frac{\operatorname{vol}(M)}{(4\pi)^{n/2}\Gamma(\frac{n}{2}+1)}\lambda^n + o(\lambda^n), \quad \lambda \rightarrow \infty.$$

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This was first proved by Weyl [We1] for a bounded domain  $\Omega \subset \mathbb{R}^3$ . Written in a slightly different form it is known in physics as the Rayleigh-Jeans law. Raleigh [Ra] derived it for a cube. Garding [Ga] proved Weyl's law for a general elliptic operator on a domain in  $\mathbb{R}^n$ . For a closed Riemannian manifold (1.2) was proved by Minakshisundaram and Pleijel [MP].

Formula (1.2) does not say very much about the finer structure of the eigenvalue distribution. The basic question is the estimation of the remainder term

$$R(\lambda) := N(\lambda) - \frac{\text{vol}(M)}{(4\pi)^{n/2} \Gamma(\frac{n}{2} + 1)} \lambda^n.$$

That this is a deep problem shows the following example. Consider the flat 2-dimensional torus  $T = \mathbb{R}^2/(2\pi\mathbb{Z})^2$ . Then the eigenvalues of the flat Laplacian are  $\lambda_{m,n} := m^2 + n^2$ ,  $m, n \in \mathbb{Z}$  and the counting function equals

$$N(\lambda) = \#\{(m, n) \in \mathbb{Z}^2 : \sqrt{m^2 + n^2} \leq \lambda\}.$$

Thus  $N(\lambda)$  is the number of lattice points in the circle of radius  $\lambda$ . An elementary packing argument, attributed to Gauss, gives

$$N(\lambda) = \pi\lambda^2 + O(\lambda),$$

and the circle problem is to find the best exponent  $\mu$  such that

$$N(\lambda) = \pi\lambda^2 + O_\varepsilon(\lambda^{\mu+\varepsilon}), \quad \forall \varepsilon > 0.$$

The conjecture of Hardy is  $\mu = 1/2$ . The first nontrivial result is due to Sierpinski who showed that one can take  $\mu = 2/3$ . Currently the best known result is  $\mu = 131/208 \approx 0.629$  which is due to Huxley. Levitan [Le] has shown that for a domain in  $\mathbb{R}^n$  the remainder term is of order  $O(\lambda^{n-1})$ .

For a closed Riemannian manifold, Avakumović [Av] proved the Weyl estimate with optimal remainder term:

$$(1.3) \quad N(\lambda) = \frac{\text{vol}(M)}{(4\pi)^{n/2} \Gamma(\frac{n}{2} + 1)} \lambda^n + O(\lambda^{n-1}), \quad \lambda \rightarrow \infty.$$

This result was extended to more general, and higher order operators by Hörmander [Ho]. As shown by Avakumović the bound  $O(\lambda^{n-1})$  of the remainder term is optimal for the sphere. On the other hand, under certain assumption on the geodesic flow, the estimate can be slightly improved. Let  $S^*M$  be the unit cotangent bundle and let  $\Phi_t$  be the geodesic flow. Suppose that the set of  $(x, \xi) \in S^*M$  such that  $\Phi_t$  has a contact of infinite order with the identity at  $(x, \xi)$  for some  $t \neq 0$ , has measure zero in  $S^*M$ . Then Duistermaat and Guillemin [DG] proved that the remainder term satisfies  $R(\lambda) = o(\lambda^{n-1})$ . This is a slight improvement over (1.3).

In [We3] Weyl formulated a conjecture which claims the existence of a second term in the asymptotic expansion for a bounded domain  $\Omega \subset \mathbb{R}^3$ , namely he predicted that

$$N(\lambda) = \frac{\text{vol}(\Omega)}{6\pi^2} \lambda^3 - \frac{\text{vol}(\partial\Omega)}{16\pi} \lambda^2 + o(\lambda^2).$$

This was proved for manifolds with boundary under a certain condition on the periodic billiard trajectories, by Ivrii [Iv] and Melrose [Me].

The purpose of this paper is to discuss Weyl's law in the context of locally symmetric spaces  $\Gamma \backslash S$  of finite volume and non-compact type. Here  $S = G/K$  is a Riemannian symmetric space, where  $G$  is a real semi-simple Lie group of non-compact type, and  $K$  a maximal compact subgroup of  $G$ . Moreover  $\Gamma$  is a lattice in  $G$ , i.e., a discrete subgroup of finite covolume. Of particular interest are arithmetic subgroups such as the principal congruence subgroup  $\Gamma(N)$  of  $\mathrm{SL}(2, \mathbb{Z})$  of level  $N \in \mathbb{N}$ . Spectral theory of the Laplacian on arithmetic quotients  $\Gamma \backslash S$  is intimately related with the theory of automorphic forms. In fact, for a symmetric space  $S$  it is more natural and important to consider not only the Laplacian, but the whole algebra  $\mathcal{D}(S)$  of  $G$ -invariant differential operators on  $S$ . It is known that  $\mathcal{D}(S)$  is a finitely generated commutative algebra [He]. Therefore, it makes sense to study the joint spectral decomposition of  $\mathcal{D}(S)$ . Square integrable joint eigenfunctions of  $\mathcal{D}(S)$  are examples of automorphic forms. Among them are the cusp forms which satisfy additional decay conditions. Cusps forms are the building blocks of the theory of automorphic forms and, according to deep and far-reaching conjectures of Langlands [La2], are expected to provide important relations between harmonic analysis and number theory.

Let  $G = NAK$  be the Iwasawa decomposition of  $G$  and let  $\mathfrak{a}$  be the Lie algebra of  $A$ . If  $\Gamma \backslash S$  is compact, the spectrum of  $\mathcal{D}(S)$  in  $L^2(\Gamma \backslash S)$  is a discrete subset of the complexification  $\mathfrak{a}_{\mathbb{C}}^*$  of  $\mathfrak{a}^*$ . It has been studied by Duistermaat, Kolk, and Varadarajan in [DKV]. The method is based on the Selberg trace formula. The results are more refined statements about the distribution of the spectrum than just the Weyl law. For example, one gets estimations for the distribution of the tempered and the complementary spectrum. We will review briefly these results in section 2.

If  $\Gamma \backslash S$  is non-compact, which is the case for many important arithmetic groups, the Laplacian has a large continuous spectrum which can be described in terms of Eisenstein series [La1]. Therefore, it is not obvious that the Laplacian has any eigenvalue  $\lambda > 0$ , and an important problem in the theory of automorphic forms is the existence and construction of cusp forms for a given lattice  $\Gamma$ . This is where the Weyl law comes into play. Let  $\mathbb{H}$  be the upper half-plane. Recall that  $\mathrm{SL}(2, \mathbb{R})$  acts on  $\mathbb{H}$  by fractional linear transformations. Using his trace formula [Se2], Selberg established the following version of Weyl's law for an arbitrary lattice  $\Gamma$  in  $\mathrm{SL}(2, \mathbb{R})$

$$(1.4) \quad N_{\Gamma}(\lambda) + M_{\Gamma}(\lambda) \sim \frac{\mathrm{Area}(\Gamma \backslash \mathbb{H})}{4\pi} \lambda^2, \quad \lambda \rightarrow \infty$$

[Se2, p. 668]. Here  $N_{\Gamma}(\lambda)$  is the counting function of the eigenvalues and  $M_{\Gamma}(\lambda)$  is the winding number of the determinant  $\phi(1/2 + ir)$  of the scattering matrix which is given by the constant Fourier coefficients of the Eisenstein series (see section 4). In general, the two functions on the left can not be estimated separately. However, for congruence groups like  $\Gamma(N)$ , the meromorphic function  $\phi(s)$  can be expressed in terms of well-known functions of analytic number theory. In this case, it is possible to show that the growth of  $M_{\Gamma}(\lambda)$

is of lower order which implies Weyl's law for the counting function of the eigenvalues [Se2, p.668]. Especially it follows that Maass cusp forms exist in abundance for congruence groups. On the other hand, there are indications [PS1], [PS2] that the existence of many cusp forms may be restricted to arithmetic groups. This will be discussed in detail in section 4.

In section 5 we discuss the general case of a non-compact arithmetic quotient  $\Gamma \backslash S$ . There has been some recent progress with the spectral problems discussed above. Lindenstrauss and Venkatesh [LV] established Weyl's law without remainder term for congruence subgroups of a split adjoint semi-simple group  $\mathbf{G}$ . In [Mu3] this had been proved for congruence subgroups of  $\mathrm{SL}(n)$  and for the Bochner-Laplace operator acting in sections of a locally homogeneous vector bundle over  $S_n = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ . For congruence subgroups of  $\mathbf{G} = \mathrm{SL}(n)$ , an estimation of the remainder term in Weyl's law has been established by E. Lapid and the author in [LM]. Using the approach of [DKV] combined with the Arthur trace formula, the results of [DKV] have been extended in [LM] to the cuspidal spectrum of  $\mathcal{D}(S_n)$ .

## 2. COMPACT LOCALLY SYMMETRIC SPACES

In this section we review Hörmanders method of the derivation of Weyl's law with remainder term for the Laplacian  $\Delta$  of a closed Riemannian manifold  $M$  of dimension  $n$ . Then we will discuss the results of [DKV] concerning spectral asymptotics for compact locally symmetric manifolds.

The method of Hörmander [Ho] to estimate the remainder term is based on the study of the kernel of  $e^{-it\sqrt{\Delta}}$ . The main point is the construction of a good approximate fundamental solution to the wave equation by means of the theory of Fourier integral operators and the analysis of the singularities of its trace

$$\mathrm{Tr} e^{-it\sqrt{\Delta}} = \sum_j e^{-it\sqrt{\lambda_j}},$$

which is well-defined as a distribution. The analysis of Hörmander of the "big" singularity of  $\mathrm{Tr} e^{-it\sqrt{\Delta}}$  at  $t = 0$  leads to the following key result [DG, (2.16)]. Let  $\mu_j := \sqrt{\lambda_j}$ ,  $j \in \mathbb{N}$ . There exist  $c_j \in \mathbb{R}$ ,  $j = 0, \dots, n-1$ , and  $\varepsilon > 0$  such that for every  $h \in \mathcal{S}(\mathbb{R})$  with  $\mathrm{supp} \hat{h} \subset [-\varepsilon, \varepsilon]$  and  $\hat{h} \equiv 1$  in a neighborhood of 0 one has

$$(2.1) \quad \sum_j h(\mu - \mu_j) \sim (2\pi)^{-n} \sum_{k=0}^{n-1} c_k \mu^{n-1-k}, \quad \mu \rightarrow \infty,$$

and rapidly decreasing as  $\mu \rightarrow -\infty$ . The constants  $c_k$  are of the form

$$c_k = \int_M \omega_k,$$

where the  $\omega_k$ 's are real valued smooth densities on  $M$  canonically associated to the Riemannian metric of  $M$ . Especially

$$c_0 = \text{vol}(S^*M), \quad c_1 = (1 - n) \int_{S^*M} \text{sub } \Delta,$$

where  $S^*M$  is the unit co-tangent bundle, and  $\text{sub } \Delta$  denotes the subprincipal symbol of  $\Delta$ . Consideration of the top term in (2.1) leads to the basic estimates for the eigenvalues.

If  $M = \Gamma \backslash G/K$  is a locally symmetric manifold, the Selberg trace formula can be used to replace (2.1) by an exact formula [DKV]. Actually, if the rank of  $M$  is bigger than 1, the spectrum is multidimensional. Then the Selberg trace formula gives more refined information.

As example, we consider a compact hyperbolic surface  $M = \Gamma \backslash \mathbb{H}$ , where  $\Gamma \subset \text{PSL}(2, \mathbb{R})$  is a discrete, torsion-free, co-compact subgroup. Let  $\Delta$  be the hyperbolic Laplace operator which is given by

$$(2.2) \quad \Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right), \quad z = x + iy.$$

Write the eigenvalues  $\lambda_j$  of  $\Delta$  as

$$\lambda_j = \frac{1}{4} + r_j^2,$$

where  $r_j \in \mathbb{C}$  and  $\arg(r_j) \in \{0, \pi/2\}$ . Let  $h$  be an analytic function in a strip  $|\text{Im}(z)| \leq \frac{1}{2} + \delta$ ,  $\delta > 0$ , such that

$$(2.3) \quad h(z) = h(-z), \quad |h(z)| \leq C(1 + |z|)^{-2-\delta}.$$

Let

$$g(u) = \frac{1}{2\pi} \int_{\mathbb{R}} h(r) e^{iru} dr.$$

Given  $\gamma \in \Gamma$  denote by  $\{\gamma\}_\Gamma$  its  $\Gamma$ -conjugacy class. Since  $\Gamma$  is co-compact, each  $\gamma \neq e$  is hyperbolic. Each hyperbolic element  $\gamma$  is the power of a primitive hyperbolic element  $\gamma_0$ . A hyperbolic conjugacy class determines a closed geodesic  $\tau_\gamma$  of  $\Gamma \backslash \mathbb{H}$ . Let  $l(\gamma)$  denote the length of  $\tau_\gamma$ . Then the Selberg trace formula [Se1] is the following identity:

$$(2.4) \quad \sum_{j=0}^{\infty} h(r_j) = \frac{\text{Area}(\Gamma \backslash \mathbb{H})}{4\pi} \int_{\mathbb{R}} h(r) r \tanh(\pi r) dr + \sum_{\{\gamma\}_\Gamma \neq e} \frac{l(\gamma_0)}{2 \sinh\left(\frac{l(\gamma)}{2}\right)} g(l(\gamma)).$$

Now let  $g \in C_c^\infty(\mathbb{R})$  and  $h(z) = \int_{\mathbb{R}} g(r) e^{-irz} dr$ . Then  $h$  is entire and rapidly decreasing in each strip  $|\text{Im}(z)| \leq c$ ,  $c > 0$ . Let  $t \in \mathbb{R}$  and set

$$h_t(z) = h(t - z) + h(t + z).$$

Then  $h_t$  is entire and satisfies (2.3). Note that  $\hat{h}_t(r) = e^{-itr}g(r) + e^{itr}g(-r)$ . We symmetrize the spectrum by  $r_{-j} := -r_j$ ,  $j \in \mathbb{N}$ . Then by (2.4) we get

$$(2.5) \quad \sum_{j=-\infty}^{\infty} h(t - r_j) = \frac{\text{Area}(\Gamma \backslash \mathbb{H})}{2\pi} \int_{\mathbb{R}} h(t - r) r \tanh(\pi r) dr \\ + \sum_{\{\gamma\}_{\Gamma} \neq e} \frac{l(\gamma_0)}{2 \sinh\left(\frac{l(\gamma)}{2}\right)} (e^{-itl(\gamma)} g(l(\gamma)) + e^{itl(\gamma)} g(-l(\gamma))).$$

Let  $\varepsilon > 0$  be such that  $l(\gamma) > \varepsilon$  for all hyperbolic conjugacy classes  $\{\gamma\}_{\Gamma}$ . The following lemma is an immediate consequence of (2.5).

**Lemma 2.1.** *Let  $g \in C_c^\infty(\mathbb{R})$  such that  $\text{supp } g \subset (-\varepsilon, \varepsilon)$ . Let  $h(z) = \int_{\mathbb{R}} g(r) e^{-irz} dr$ . Then for all  $t \in \mathbb{R}$  we have*

$$(2.6) \quad \sum_{j=-\infty}^{\infty} h(t - r_j) = \frac{\text{Area}(\Gamma \backslash \mathbb{H})}{2\pi} \int_{\mathbb{R}} h(t - r) r \tanh(\pi r) dr.$$

Changing variables in the integral on the right and using that

$$\tanh(\pi(r+t)) = 1 - \frac{2e^{-2\pi(r+t)}}{1 + e^{-2\pi(r+t)}} = -1 + \frac{2e^{2\pi(r+t)}}{1 + e^{2\pi(r+t)}},$$

we obtain the following asymptotic expansion

$$(2.7) \quad \sum_{j=-\infty}^{\infty} h(t - r_j) = \frac{\text{Area}(\Gamma \backslash \mathbb{H})}{2\pi} \left( |t| \int_{\mathbb{R}} h(r) dr - \text{sign } t \int_{\mathbb{R}} h(r) r dr \right) + O(e^{-2\pi|t|}),$$

as  $|t| \rightarrow \infty$ . If  $h$  is even, the second term vanishes and the asymptotic expansion is related to (2.1). The asymptotic expansion (2.7) can be used to derive estimates for the number of eigenvalues near a given point  $\mu \in \mathbb{R}$ .

**Lemma 2.2.** *For every  $a > 0$  there exists  $C > 0$  such that*

$$\#\{j: |r_j - \mu| \leq a\} \leq C(1 + |\mu|)$$

for all  $\mu \in \mathbb{R}$ .

*Proof.* We proceed as in the proof of Lemma 2.3 in [DG]. As shown in the proof, there exists  $h \in \mathcal{S}(\mathbb{R})$  such that  $h \geq 0$ ,  $h > 0$  on  $[-a, a]$ ,  $\hat{h}(0) = 1$ , and  $\text{supp } \hat{h}$  is contained in any prescribed neighborhood of 0. Now observe that there are only finitely many eigenvalues  $\lambda_j = 1/4 + r_j^2$  with  $r_j \notin \mathbb{R}$ . Therefore it suffices to consider  $r_j \in \mathbb{R}$ . Let  $\mu \in \mathbb{R}$ . By (2.7) we get

$$\#\{j: |r_j - \mu| \leq a, r_j \in \mathbb{R}\} \cdot \min\{h(u): |u| \leq a\} \leq \sum_{r_j \in \mathbb{R}} h(\mu - r_j) \leq C(1 + |\mu|).$$

□

This lemma is the basis of the following auxiliary results.

**Lemma 2.3.** *For every  $h$  as above there exists  $C > 0$  such that*

$$(2.8) \quad \sum_{|r_j| \leq \lambda} \left| \int_{\mathbb{R} - [-\lambda, \lambda]} h(t - r_j) dt \right| \leq C\lambda, \quad \sum_{|r_j| > \lambda} \left| \int_{-\lambda}^{\lambda} h(t - r_j) dt \right| \leq C\lambda,$$

for all  $\lambda \geq 1$ .

*Proof.* Since  $h$  is rapidly decreasing, there exists  $C > 0$  such that  $|h(t)| \leq C(1 + |t|)^{-4}$ ,  $t \in \mathbb{R}$ . Let  $[\lambda]$  be the largest integer  $\leq \lambda$ . Then we get

$$\begin{aligned} \sum_{|r_j| \leq \lambda} \left| \int_{-\lambda}^{\infty} h(t - r_j) dt \right| &\leq \sum_{|r_j| \leq \lambda} \int_{\lambda - r_j}^{\infty} |h(t)| dt \leq C \sum_{|r_j| \leq \lambda} \frac{1}{(1 + \lambda - r_j)^3} \\ &= \sum_{k=-[\lambda]}^{[\lambda]-1} \sum_{k \leq r_j \leq k+1} \frac{1}{(1 + \lambda - r_j)^3} \leq \sum_{k=-[\lambda]}^{[\lambda]-1} \frac{\#\{j: |r_j - k| \leq 1\}}{(\lambda - k)^3}, \end{aligned}$$

and by Lemma 2.2 the right hand side is bounded by  $C\lambda$  for  $\lambda \geq 1$ . Similarly we get

$$\sum_{|r_j| \leq \lambda} \left| \int_{-\infty}^{-\lambda} h(t - r_j) dt \right| \leq C_2\lambda.$$

The second series can be treated in the same way. □

**Lemma 2.4.** *Let  $h$  be as in Lemma 2.1 and such that  $\hat{h}(0) = 1$ . Then*

$$(2.9) \quad \int_{-\lambda}^{\lambda} \sum_{j=-\infty}^{\infty} h(t - r_j) dt = \frac{\text{Area}(\Gamma \backslash \mathbb{H})}{2\pi} \lambda^2 + O(\lambda)$$

as  $\lambda \rightarrow \infty$ .

*Proof.* To prove the lemma, we integrate (2.6) and determine the asymptotic behavior of the integral on the right. Let  $p(r)$  be a continuous function on  $\mathbb{R}$  such that  $|p(r)| \leq C(1 + |r|)$  and  $p(r) = p(-r)$ . Changing the order of integration and using that  $\int_{\mathbb{R}} h(t - r) dt = \hat{h}(0) = 1$ , we get

$$\begin{aligned} \int_{-\lambda}^{\lambda} \int_{\mathbb{R}} h(t - r) p(r) dr dt &= \int_{-\lambda}^{\lambda} p(r) dr - \int_{-\lambda}^{\lambda} \left( \int_{\mathbb{R} - [-\lambda, \lambda]} h(t - r) dt \right) p(r) dr \\ &\quad + \int_{\mathbb{R} - [-\lambda, \lambda]} \left( \int_{-\lambda}^{\lambda} h(t - r) dt \right) p(r) dr. \end{aligned}$$

Let  $C_1 > 0$  be such that  $|h(r)| \leq C_1(1 + |r|)^{-3}$ ,  $r \in \mathbb{R}$ . Then the second and the third integral can be estimated by  $C(1 + \lambda)$ . Thus we get

$$(2.10) \quad \int_{-\lambda}^{\lambda} \int_{\mathbb{R}} h(t - r) p(r) dr dt = \int_{-\lambda}^{\lambda} p(r) dr + O(\lambda), \quad \lambda \rightarrow \infty.$$

If we apply (2.10) to  $p(r) = r \tanh(\pi r)$ , we obtain

$$(2.11) \quad \int_{-\lambda}^{\lambda} \int_{\mathbb{R}} h(t-r) r \tanh(\pi r) dr dt = \lambda^2 + O(\lambda).$$

This proves the lemma.  $\square$

We are now ready to prove Weyl's law. We choose  $h$  such that  $\hat{h}$  has sufficiently small support and  $\hat{h}(0) = 1$ . Then

$$\begin{aligned} \int_{-\lambda}^{\lambda} \sum_{j=-\infty}^{\infty} h(t-r_j) dt &= \sum_{|r_j| \leq \lambda} \int_{\mathbb{R}} h(t-r_j) dt - \sum_{|r_j| \leq \lambda} \int_{\mathbb{R}-[-\lambda, \lambda]} h(t-r_j) dt \\ &\quad + \sum_{|r_j| > \lambda} \int_{-\lambda}^{\lambda} h(t-r_j) dt. \end{aligned}$$

Using that  $\int_{\mathbb{R}} h(t-r) dt = \hat{h}(0) = 1$ , we get

$$\begin{aligned} 2N_{\Gamma}(\lambda) &= \int_{-\lambda}^{\lambda} \sum_j h(t-r_j) dt + \sum_{|r_j| \leq \lambda} \int_{\mathbb{R}-[-\lambda, \lambda]} h(t-r_j) dt \\ &\quad - \sum_{|r_j| > \lambda} \int_{-\lambda}^{\lambda} h(t-r_j) dt. \end{aligned}$$

By Lemmas 2.3 and 2.4 we obtain

$$(2.12) \quad N_{\Gamma}(\lambda) = \frac{\text{Area}(\Gamma \backslash \mathbb{H})}{4\pi} \lambda^2 + O(\lambda).$$

We turn now to an arbitrary Riemannian symmetric space  $S = G/K$  of non-compact type and we review the main results of [DKV]. The group of motions  $G$  of  $S$  is a semi-simple Lie group of non-compact type with finite center and  $K$  is a maximal compact subgroup of  $G$ . The Laplacian  $\Delta$  of  $S$  is a  $G$ -invariant differential operator on  $S$ , i.e.,  $\Delta$  commutes with the left translations  $L_g$ ,  $g \in G$ . Besides of  $\Delta$  we need to consider the ring  $\mathcal{D}(S)$  of all invariant differential operators on  $S$ . It is well-known that  $\mathcal{D}(S)$  is commutative and finitely generated. Its structure can be described as follows. Let  $G = NAK$  be the Iwasawa decomposition of  $G$ ,  $W$  the Weyl group of  $(G, A)$  and  $\mathfrak{a}$  be the Lie algebra of  $A$ . Let  $S(\mathfrak{a}_{\mathbb{C}})$  be the symmetric algebra of the complexification  $\mathfrak{a}_{\mathbb{C}} = \mathfrak{a} \otimes \mathbb{C}$  of  $\mathfrak{a}$  and let  $S(\mathfrak{a}_{\mathbb{C}})^W$  be the subspace of Weyl group invariants in  $S(\mathfrak{a}_{\mathbb{C}})$ . Then by a theorem of Harish-Chandra [He, Ch. X, Theorem 6.15] there is a canonical isomorphism

$$(2.13) \quad \mu: \mathcal{D}(S) \cong S(\mathfrak{a}_{\mathbb{C}})^W.$$

This shows that  $\mathcal{D}(S)$  is commutative. The minimal number of generators equals the rank of  $S$  which is  $\dim \mathfrak{a}$  [He, Ch.X, §6.3]. Let  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ . Then by (2.13),  $\lambda$  determines an character

$$\chi_{\lambda}: \mathcal{D}(S) \rightarrow \mathbb{C}$$



and  $\chi_\lambda = \chi_{\lambda'}$  if and only if  $\lambda$  and  $\lambda'$  are in the same  $W$ -orbit. Since  $S(\mathfrak{a}_\mathbb{C})$  is integral over  $S(\mathfrak{a}_\mathbb{C})^W$  [He, Ch. X, Lemma 6.9], each character of  $\mathcal{D}(S)$  is of the form  $\chi_\lambda$  for some  $\lambda \in \mathfrak{a}_\mathbb{C}^*$ . Thus the characters of  $\mathcal{D}(S)$  are parametrized by  $\mathfrak{a}_\mathbb{C}^*/W$ .

Let  $\Gamma \subset G$  be a discrete, torsion-free, co-compact subgroup of  $G$ . Then  $\Gamma$  acts properly discontinuously on  $S$  without fixed points and the quotient  $M = \Gamma \backslash S$  is a locally symmetric manifold which is equipped with the metric induced from the invariant metric of  $S$ . Then each  $D \in \mathcal{D}(S)$  descends to a differential operator

$$D: C^\infty(\Gamma \backslash S) \rightarrow C^\infty(\Gamma \backslash S).$$

Let  $\mathcal{E} \subset C^\infty(\Gamma \backslash S)$  be an eigenspace of the Laplace operator. Then  $\mathcal{E}$  is a finite-dimensional vector space which is invariant under  $D \in \mathcal{D}(S)$ . For each  $D \in \mathcal{D}(S)$ , the formal adjoint  $D^*$  of  $D$  also belongs to  $\mathcal{D}(S)$ . Thus we get a representation

$$\rho: \mathcal{D}(S) \rightarrow \text{End}(\mathcal{E})$$

by commuting normal operators. Therefore,  $\mathcal{E}$  decomposes into the direct sum of joint eigenspaces of  $\mathcal{D}(S)$ . Given  $\lambda \in \mathfrak{a}_\mathbb{C}^*/W$ , let

$$\mathcal{E}(\lambda) = \{\varphi \in C^\infty(\Gamma \backslash S): D\varphi = \chi_\lambda(D)\varphi, D \in \mathcal{D}(S)\}.$$

Let  $m(\lambda) = \dim \mathcal{E}(\lambda)$ . Then the spectrum  $\Lambda(\Gamma)$  of  $\Gamma \backslash S$  is defined to be

$$\Lambda(\Gamma) = \{\lambda \in \mathfrak{a}_\mathbb{C}^*/W: m(\lambda) > 0\},$$

and we get an orthogonal direct sum decomposition

$$L^2(\Gamma \backslash S) = \bigoplus_{\lambda \in \Lambda(\Gamma)} \mathcal{E}(\lambda).$$

If we pick a fundamental domain for  $W$ , we may regard  $\Lambda(\Gamma)$  as a subset of  $\mathfrak{a}_\mathbb{C}^*$ . If  $\text{rank}(S) > 1$ , then  $\Lambda(\Gamma)$  is multidimensional. Again the distribution of  $\Lambda(\Gamma)$  is studied using the Selberg trace formula [Se1]. To describe it we need to introduce some notation. Let  $C_c^\infty(G//K)$  be the subspace of all  $f \in C_c^\infty(G)$  which are  $K$ -bi-invariant. Let

$$\mathcal{A}: C_c^\infty(G//K) \rightarrow C_c^\infty(A)^W$$

be the Abel transform which is defined by

$$\mathcal{A}(f)(a) = \delta(a)^{1/2} \int_N f(an) dn, \quad a \in A,$$

where  $\delta$  is the modulus function of the minimal parabolic subgroup  $P = NA$ . Given  $h \in C_c^\infty(A)^W$ , let

$$\hat{h}(\lambda) = \int_A h(a) e^{\langle \lambda, H(a) \rangle} da.$$

Let  $\beta(i\lambda)$ ,  $\lambda \in \mathfrak{a}^*$ , be the Plancherel density. Then the Selberg trace formula is the following identity

$$(2.14) \quad \sum_{\lambda \in \Lambda(\Gamma)} m(\lambda) \hat{h}(\lambda) = \frac{\text{vol}(\Gamma \backslash G)}{|W|} \int_{\mathfrak{a}^*} \hat{h}(\lambda) \beta(i\lambda) d\lambda \\ + \sum_{[\gamma]_{\Gamma} \neq e} \text{vol}(\Gamma_{\gamma} \backslash G_{\gamma}) \int_{G_{\gamma} \backslash G} \mathcal{A}^{-1}(h)(x^{-1}\gamma x) d_{\gamma} \bar{x}.$$

This is still not the final form of the Selberg trace formula. The distributions

$$(2.15) \quad J_{\gamma}(f) = \text{vol}(\Gamma_{\gamma} \backslash G_{\gamma}) \int_{G_{\gamma} \backslash G} f(x^{-1}\gamma x) d_{\gamma} \bar{x}, \quad f \in C_c^{\infty}(G),$$

are invariant distribution on  $G$  and can be computed using Harish-Chandra's Fourier inversion formula. This brings (2.14) into a form which is similar to (2.4). For the present purpose, however, it suffices to work with (2.14). Since for  $\gamma \neq e$ , the conjugacy class of  $\gamma$  in  $G$  is closed and does not intersect  $K$ , there exists an open neighborhood  $V$  of 1 in  $A$  satisfying  $V = V^{-1}$ ,  $V$  is invariant under  $W$ , and  $J_{\gamma}(\mathcal{A}^{-1}(h)) = 0$  for all  $h \in C_c^{\infty}(V)$  [DKV, Proposition 3.8]. Thus we get

$$(2.16) \quad \sum_{\lambda \in \Lambda(\Gamma)} m(\lambda) \hat{h}(\lambda) = \frac{\text{vol}(\Gamma \backslash G)}{|W|} \int_{\mathfrak{a}^*} \hat{h}(\lambda) \beta(i\lambda) d\lambda$$

for all  $h \in C_c^{\infty}(V)$ . One can now proceed as in the case of the upper half-plane. The basic step is again to estimate the number of  $\lambda \in \Lambda(\Gamma)$  lying in a ball of radius  $r$  around a variable point  $\mu \in i\mathfrak{a}^*$ . This can be achieved by inserting appropriate test functions  $h$  into (2.16) [DKV, section 7]. Let

$$\Lambda_{\text{temp}}(\Gamma) = \Lambda(\Gamma) \cap i\mathfrak{a}^*, \quad \Lambda_{\text{comp}}(\Gamma) = \Lambda(\Gamma) \setminus \Lambda_{\text{temp}}(\Gamma)$$

be the tempered and complementary spectrum, respectively. Given an open bounded subset  $\Omega$  of  $\mathfrak{a}^*$  and  $t > 0$ , let

$$(2.17) \quad t\Omega := \{t\mu : \mu \in \Omega\}.$$

One of the main results of [DKV] is the following asymptotic formula for the distribution of the tempered spectrum [DKV, Theorem 8.8]

$$(2.18) \quad \sum_{\lambda \in \Lambda_{\text{temp}}(\Gamma) \cap (it\Omega)} m(\lambda) = \frac{\text{vol}(\Gamma \backslash G)}{|W|} \int_{it\Omega} \beta(i\lambda) d\lambda + O(t^{n-1}), \quad t \rightarrow \infty,$$

Note that the leading term is of order  $O(t^n)$ . The growth of the complementary spectrum is of lower order. Let  $B_t(0) \subset \mathfrak{a}_{\mathbb{C}}^*$  be the ball of radius  $t > 0$  around 0. There exists  $C > 0$  such that for all  $t \geq 1$

$$(2.19) \quad \sum_{\lambda \in \Lambda_{\text{comp}}(\Gamma) \cap B_t(0)} m(\lambda) \leq Ct^{n-2}$$

[DKV, Theorem 8.3]. The estimations (2.18) and (2.19) contain more information about the distribution of  $\Lambda(\Gamma)$  than just the Weyl law. Indeed, the eigenvalue of  $\Delta$  corresponding to  $\lambda \in \Lambda_{\text{temp}}(\Gamma)$  equals  $\|\lambda\|^2 + \|\rho\|^2$ . So if we choose  $\Omega$  in (2.18) to be the unit ball, then (2.18) together with (2.19) reduces to Weyl's law for  $\Gamma \backslash S$ .

We note that (2.18) and (2.19) can also be rephrased in terms of representation theory. Let  $R$  be the right regular representation of  $G$  in  $L^2(\Gamma \backslash G)$  defined by

$$(R(g_1)f)(g_2) = f(g_2g_1), \quad f \in L^2(\Gamma \backslash G), \quad g_1, g_2 \in G.$$

Let  $\hat{G}$  be the unitary dual of  $G$ , i.e., the set of equivalence classes of irreducible unitary representations of  $G$ . Since  $\Gamma \backslash G$  is compact, it is well known that  $R$  decomposes into direct sum of irreducible unitary representations of  $G$ . Given  $\pi \in \hat{G}$ , let  $m(\pi)$  be the multiplicity with which  $\pi$  occurs in  $R$ . Let  $\mathcal{H}_\pi$  denote the Hilbert space in which  $\pi$  acts. Then

$$L^2(\Gamma \backslash G) \cong \bigoplus_{\pi \in \hat{G}} m(\pi) \mathcal{H}_\pi.$$

Now observe that  $L^2(\Gamma \backslash S) = L^2(\Gamma \backslash G)^K$ . Let  $\mathcal{H}_\pi^K$  denote the subspace of  $K$ -fixed vectors in  $\mathcal{H}_\pi$ . Then

$$L^2(\Gamma \backslash S) \cong \bigoplus_{\pi \in \hat{G}} m(\pi) \mathcal{H}_\pi^K.$$

Note that  $\dim \mathcal{H}_\pi^K \leq 1$ . Let  $\hat{G}(1) \subset \hat{G}$  be the subset of all  $\pi$  with  $\mathcal{H}_\pi^K \neq \{0\}$ . This is the spherical dual. Given  $\pi \in \hat{G}$ , let  $\lambda_\pi$  be the infinitesimal character of  $\pi$ . If  $\pi \in \hat{G}(1)$ , then  $\lambda_\pi \in \mathfrak{a}_\mathbb{C}^*/W$ . Moreover  $\pi \in \hat{G}(1)$  is tempered, if  $\pi$  is unitarily induced from the minimal parabolic subgroup  $P = NA$ . In this case we have  $\lambda_\pi \in i\mathfrak{a}^*/W$ . So (2.18) can be rewritten as

$$(2.20) \quad \sum_{\substack{\pi \in \hat{G}(1) \\ \lambda_\pi \in it\Omega}} m(\pi) = \frac{\text{vol}(\Gamma \backslash G)}{|W|} \int_{it\Omega} \beta(\lambda) d\lambda + O(t^{n-1}), \quad t \rightarrow \infty.$$

### 3. AUTOMORPHIC FORMS

The theory of automorphic forms is concerned with harmonic analysis on locally symmetric spaces  $\Gamma \backslash S$  of finite volume. Of particular interest are arithmetic groups  $\Gamma$ . This means that we consider a connected semi-simple algebraic group  $\mathbf{G}$  defined over  $\mathbb{Q}$  such that  $G = \mathbf{G}(\mathbb{R})$  and  $\Gamma$  is a subgroup of  $\mathbf{G}(\mathbb{Q})$  which is commensurable with  $\mathbf{G}(\mathbb{Z})$ , where  $\mathbf{G}(\mathbb{Z})$  is defined with respect to some embedding  $\mathbf{G} \subset \text{GL}(m)$ . The standard example is  $\mathbf{G} = \text{SL}(n)$  and  $\Gamma(N) \subset \text{SL}(n, \mathbb{Z})$  the principal congruence subgroup of level  $N$ . A basic feature of arithmetic groups is that the quotient  $\Gamma \backslash S$  has finite volume [BH]. Moreover in many important cases it is non-compact. A typical example for that is  $\Gamma(N) \backslash \text{SL}(n, \mathbb{R}) / \text{SO}(n)$ .

In this section we discuss only the case of the upper half-plane  $\mathbb{H}$  and we consider congruence subgroups of  $\text{SL}(2, \mathbb{Z})$ . For  $N \geq 1$  the principal congruence subgroup  $\Gamma(N)$  of level

$N$  is defined as

$$\Gamma(N) = \{\gamma \in \mathrm{SL}(2, \mathbb{Z}) : \gamma \equiv \mathrm{Id} \pmod{N}\}.$$

A congruence subgroup  $\Gamma$  of  $\mathrm{SL}(2, \mathbb{Z})$  is a subgroup for which there exists  $N \in \mathbb{N}$  such that  $\Gamma$  contains  $\Gamma(N)$ . An example of a congruence subgroup is the Hecke group

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

If  $\Gamma$  is torsion free, the quotient  $\Gamma \backslash \mathbb{H}$  is a finite area, non-compact, hyperbolic surface. It has a decomposition

$$(3.1) \quad \Gamma \backslash \mathbb{H} = M_0 \cup Y_1 \cup \cdots \cup Y_m,$$

into the union of a compact surface with boundary  $M_0$  and a finite number of ends  $Y_i \cong [a, \infty) \times S^1$  which are equipped with the Poincaré metric. In general,  $\Gamma \backslash \mathbb{H}$  may have a finite number of quotient singularities. The quotient  $\Gamma(N) \backslash \mathbb{H}$  is the modular surface  $X(N)$ .

Let  $\Delta$  be the hyperbolic Laplacian (2.2). A Maass automorphic form is a smooth function  $f : \mathbb{H} \rightarrow \mathbb{C}$  which satisfies

- a)  $f(\gamma z) = f(z)$ ,  $\gamma \in \Gamma$ .
- b) There exists  $\lambda \in \mathbb{C}$  such that  $\Delta f = \lambda f$ .
- c)  $f$  is slowly increasing.

Here the last condition means that there exist  $C > 0$  and  $N \in \mathbb{N}$  such that the restriction  $f_i$  of  $f$  to  $Y_i$  satisfies

$$|f_i(y, x)| \leq C y^N, \quad y \geq a, \quad i = 1, \dots, m.$$

Examples are the Eisenstein series. Let  $a_1, \dots, a_m \in \mathbb{R} \cup \{\infty\}$  be representatives of the  $\Gamma$ -conjugacy classes of parabolic fixed points of  $\Gamma$ . The  $a_i$ 's are called *cusps*. For each  $a_i$  let  $\Gamma_{a_i}$  be the stabilizer of  $a_i$  in  $\Gamma$ . Choose  $\sigma_i \in \mathrm{SL}(2, \mathbb{R})$  such that

$$\sigma_i(\infty) = a_i, \quad \sigma_i^{-1} \Gamma_{a_i} \sigma_i = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

Then the Eisenstein series  $E_i(z, s)$  associated to the cusp  $a_i$  is defined as

$$(3.2) \quad E_i(z, s) = \sum_{\gamma \in \Gamma_{a_i} \backslash \Gamma} \mathrm{Im}(\sigma_i^{-1} \gamma z)^s, \quad \mathrm{Re}(s) > 1.$$

The series converges absolutely and uniformly on compact subsets of the half-plane  $\mathrm{Re}(s) > 1$  and it satisfies the following properties.

- 1)  $E_i(\gamma z, s) = E_i(z, s)$  for all  $\gamma \in \Gamma$ .
- 2) As a function of  $s$ ,  $E_i(z, s)$  admits a meromorphic continuation to  $\mathbb{C}$  which is regular on the line  $\mathrm{Re}(s) = 1/2$ .
- 3)  $E_i(z, s)$  is a smooth function of  $z$  and satisfies  $\Delta_z E_i(z, s) = s(1-s)E_i(z, s)$ .

As example consider the modular group  $\Gamma(1)$  which has a single cusp  $\infty$ . The Eisenstein series attached to  $\infty$  is the well-known series

$$E(z, s) = \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ (m,n)=1}} \frac{y^s}{|mz + n|^{2s}}.$$

In the general case, the Eisenstein series were first studied by Selberg [Se1]. The Eisenstein series are closely related with the study of the spectral resolution of  $\Delta$ . Regarded as unbounded operator

$$\Delta: C_c^\infty(\Gamma \backslash \mathbb{H}) \rightarrow L^2(\Gamma \backslash \mathbb{H}),$$

$\Delta$  is essentially self-adjoint [Roe]. Let  $\bar{\Delta}$  be the unique self-adjoint extension of  $\Delta$ . The important new feature due to the non-compactness of  $\Gamma \backslash \mathbb{H}$  is that  $\bar{\Delta}$  has a large continuous spectrum which is governed by the Eisenstein series. The following basic result is due to Roelcke [Roe].

**Proposition 3.1.** *The spectrum of  $\bar{\Delta}$  is the union of a pure point spectrum  $\sigma_{pp}(\bar{\Delta})$  and an absolutely continuous spectrum  $\sigma_{ac}(\bar{\Delta})$ .*

1) *The pure point spectrum consists of eigenvalues  $0 = \lambda_0 < \lambda_1 \leq \dots$  of finite multiplicities with no finite points of accumulation.*

2) *The absolutely continuous spectrum equals  $[1/4, \infty)$  with multiplicity equal to the number of cusps of  $\Gamma \backslash \mathbb{H}$ .*

Of particular interest are the eigenfunctions of  $\bar{\Delta}$ . They are Maass automorphic forms. This can be seen by studying the Fourier expansion of an eigenfunction in the cusps. As an example consider  $f \in C^\infty(\Gamma_0(N) \backslash \mathbb{H})$  which satisfies

$$\Delta f = \lambda f, \quad f(z) = f(-\bar{z}), \quad \int_{\Gamma_0(N) \backslash \mathbb{H}} |f(z)|^2 dA(z) < \infty.$$

Assume that  $\lambda = 1/4 + r^2$ ,  $r \in \mathbb{R}$ . Then  $f(x + iy)$  admits a Fourier expansion w.r.t.  $x$  of the form

$$(3.3) \quad f(x + iy) = \sum_{n=1}^{\infty} a(n) \sqrt{y} K_{ir}(2\pi ny) \cos(2\pi nx),$$

where  $K_\nu(y)$  is the modified Bessel function which may be defined by

$$K_\nu(y) = \int_0^\infty e^{-y \cosh t} \cosh(\nu t) dt$$

and it satisfies

$$K_\nu''(y) + \frac{1}{y} K_\nu'(y) + \left(1 - \frac{\nu^2}{y^2}\right) K_\nu(y) = 0.$$

Now note that  $K_\nu(y) = O(e^{-cy})$  as  $y \rightarrow \infty$ . This implies that  $f$  is rapidly decreasing in the cusp  $\infty$ . A similar Fourier expansion holds in the other cusps. This implies that  $f$  is rapidly decreasing in all cusps and therefore, it is a Maass automorphic form. In fact, since the zero Fourier coefficients vanish in all cusps,  $f$  is a Maass *cuspidal form*. In general,

the space of cusp forms  $L_{\text{cus}}^2(\Gamma \backslash \mathbb{H})$  is defined as the subspace of all  $f \in L^2(\Gamma \backslash \mathbb{H})$  such that for almost all  $y \in \mathbb{R}^+$ :

$$\int_0^1 f(\sigma_k(x + iy)) dx = 0, \quad k = 1, \dots, m.$$

This is an invariant subspace of  $\bar{\Delta}$  and the restriction of  $\bar{\Delta}$  to  $L_{\text{cus}}^2(\Gamma \backslash \mathbb{H})$  has pure point spectrum, i.e.,  $L_{\text{cus}}^2(\Gamma \backslash \mathbb{H})$  is the span of square integrable eigenfunctions of  $\Delta$ . Let  $L_{\text{res}}^2(\Gamma \backslash \mathbb{H})$  be the orthogonal complement of  $L_{\text{cus}}^2(\Gamma \backslash \mathbb{H})$  in  $L^2(\Gamma \backslash \mathbb{H})$ . Thus

$$L_{pp}^2(\Gamma \backslash \mathbb{H}) = L_{\text{cus}}^2(\Gamma \backslash \mathbb{H}) \oplus L_{\text{res}}^2(\Gamma \backslash \mathbb{H}).$$

The subspace  $L_{\text{res}}^2(\Gamma \backslash \mathbb{H})$  can be described as follows. The poles of the Eisenstein series  $E_i(z, s)$  in the half-plane  $\text{Re}(s) > 1/2$  are all simple and are contained in the interval  $(1/2, 1]$ . Let  $s_0 \in (1/2, 1]$  be a pole of  $E_i(z, s)$  and put

$$\psi = \text{Res}_{s=s_0} E_i(z, s).$$

Then  $\psi$  is a square integrable eigenfunction of  $\Delta$  with eigenvalue  $\lambda = s_0(1 - s_0)$ . The set of all such residues of the Eisenstein series  $E_i(z, s)$ ,  $i = 1, \dots, m$ , spans  $L_{\text{res}}^2(\Gamma \backslash \mathbb{H})$ . This is a finite-dimensional space which is called the *residual subspace*. The corresponding eigenvalues form the *residual spectrum* of  $\bar{\Delta}$ . So we are left with the cuspidal eigenfunctions or *Maass cusp forms*. Cusp forms are the building blocks of the theory of automorphic forms. They play an important role in number theory. To illustrate this consider an even Maass cusp form  $f$  for  $\Gamma(1)$  with eigenvalue  $\lambda = 1/4 + r^2$ ,  $r \in \mathbb{R}$ . Let  $a(n)$ ,  $n \in \mathbb{N}$ , be the Fourier coefficients of  $f$  given by (3.3). Put

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \quad \text{Re}(s) > 1.$$

This Dirichlet series converges absolutely and uniformly in the half-plane  $\text{Re}(s) > 1$ . Let

$$(3.4) \quad \Lambda(s, f) = \pi^{-s} \Gamma\left(\frac{s + ir}{2}\right) \Gamma\left(\frac{s - ir}{2}\right) L(s, f).$$

Then the modularity of  $f$  implies that  $\Lambda(s, f)$  has an analytic continuation to the whole complex plane and satisfies the functional equation

$$\Lambda(s, f) = \Lambda(1 - s, f)$$

[Bu, Proposition 1.9.1]. Under additional assumptions on  $f$ , the Dirichlet series  $L(s, f)$  is also an Euler product. This is related to the arithmetic nature of the groups  $\Gamma(N)$ . The surfaces  $X(N)$  carry a family of algebraically defined operators  $T_n$ , the so called *Hecke operators*, which for  $(n, N) = 1$  are defined by

$$T_n f(z) = \frac{1}{\sqrt{n}} \sum_{\substack{ad=n \\ b \pmod{d}}} f\left(\frac{az + b}{d}\right).$$

These are closely related to the cosets of the finite index subgroups

$$\begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} \Gamma(N) \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}^{-1} \cap \Gamma(N)$$

of  $\Gamma(N)$ . Each  $T_n$  defines a linear transformation of  $L^2(X(N))$ . The  $T_n$ ,  $n \in \mathbb{N}$ , are a commuting family of normal operators which also commute with  $\Delta$ . Therefore, each  $T_n$  leaves the eigenspaces of  $\Delta$  invariant. So we may assume that  $f$  is a common eigenfunction of  $\Delta$  and  $T_n$ ,  $n \in \mathbb{N}$ :

$$\Delta f = (1/4 + r^2)f, \quad T_n f = \lambda(n)f.$$

If  $f \neq 0$ , then  $a(1) \neq 0$ . So we can normalize  $f$  such that  $a(1) = 1$ . Then it follows that  $a(n) = \lambda(n)$  and the Fourier coefficients satisfy the following multiplicative relations

$$a(m)a(n) = \sum_{d|(m,n)} a\left(\frac{mn}{d^2}\right).$$

This implies that  $L(s, f)$  is an Euler product

$$(3.5) \quad L(s, f) = \sum_{n=1}^{\infty} a(n)n^{-s} = \prod_p (1 - a(p)p^{-s} + p^{-2s})^{-1},$$

which converges for  $\text{Re}(s) > 1$ .  $L(s, f)$  is the basic example of an automorphic  $L$ -function. It is convenient to write this Euler product in a different way. Introduce roots  $\alpha_p, \beta_p$  by

$$\alpha_p \beta_p = 1, \quad \alpha_p + \beta_p = a(p).$$

Let

$$A_p = \begin{pmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{pmatrix}.$$

Then

$$L(s, f) = \prod_p \det (\text{Id} - A_p p^{-s})^{-1}.$$

Now let  $\rho: \text{GL}(2, \mathbb{C}) \rightarrow \text{GL}(N, \mathbb{C})$  be a representation. Then we can form a new Euler product by

$$L(s, f, \rho) = \prod_p \det (\text{Id} - \rho(A_p) p^{-s})^{-1},$$

which converges in some half-plane. It is part of the general conjectures of Langlands [La2] that each of these Euler products admits a meromorphic extension to  $\mathbb{C}$  and satisfies a functional equation. The construction of Euler products for Maass cusp forms can be extended to other groups  $\mathbf{G}$ , in particular to cusp forms on  $\text{GL}(n)$ . It is also conjectured that  $L(s, f, \rho)$  is an automorphic  $L$ -function of an automorphic form on some  $\text{GL}(n)$ . This is part of the functoriality principle of Langlands. Furthermore, all  $L$ -functions that occur in number theory and algebraic geometry are expected to be automorphic  $L$ -functions. This is one of the main reasons for the interest in the study of cusp forms. Other applications are discussed in [Sa1].

## 4. THE WEYL LAW AND EXISTENCE OF CUSP FORMS

Since  $\Gamma(N)\backslash\mathbb{H}$  is not compact, it is not clear that there exist any eigenvalues  $\lambda > 0$ . By Proposition 3.1 the continuous spectrum of  $\bar{\Delta}$  equals  $[1/4, \infty)$ . Thus all eigenvalues  $\lambda \geq 1/4$  are embedded in the continuous spectrum. It is well-known in mathematical physics, that embedded eigenvalues are unstable under perturbations and therefore, are difficult to study.

One of the basic tools to study the cuspidal spectrum is the Selberg trace formula [Se2]. The new terms in the trace formula, which are due to the non-compactness of  $\Gamma\backslash\mathbb{H}$  arise from the parabolic conjugacy classes in  $\Gamma$  and the Eisenstein series. The contribution of the Eisenstein series is given by their zeroth Fourier coefficients of the Fourier expansion in the cusps. The zeroth Fourier coefficient of the Eisenstein series  $E_k(z, s)$  in the cusp  $a_l$  is given by

$$\int_0^1 E_k(\sigma_l(x + iy), s) dx = y^s + C_{kl}(s)y^{1-s},$$

where  $C_{kl}(s)$  is a meromorphic function of  $s \in \mathbb{C}$ . Put

$$C(s) := (C_{kl}(s))_{k,l=1}^m.$$

This is the so called scattering matrix. Let

$$\phi(s) := \det C(s).$$

Let the notation be as in (2.4) and assume that  $\Gamma$  has no torsion. Then the trace formula is the following identity.

$$\begin{aligned} \sum_j h(r_j) &= \frac{\text{Area}(\Gamma\backslash\mathbb{H})}{4\pi} \int_{\mathbb{R}} h(r) r \tanh(\pi r) dr + \sum_{\{\gamma\}_r} \frac{l(\gamma_0)}{2 \sinh\left(\frac{l(\gamma)}{2}\right)} g(l(\gamma)) \\ (4.1) \quad &+ \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \frac{\phi'}{\phi}(1/2 + ir) dr - \frac{1}{4} \phi(1/2) h(0) \\ &- \frac{m}{2\pi} \int_{-\infty}^{\infty} h(r) \frac{\Gamma'}{\Gamma}(1 + ir) dr + \frac{m}{4} h(0) - m \ln 2 g(0). \end{aligned}$$

The trace formula holds for every discrete subgroup  $\Gamma \subset \text{SL}(2, \mathbb{R})$  with finite coarea. In analogy to the counting function of the eigenvalues we introduce the winding number

$$M_{\Gamma}(\lambda) = -\frac{1}{4\pi} \int_{-\lambda}^{\lambda} \frac{\phi'}{\phi}(1/2 + ir) dr$$

which measures the continuous spectrum. Using the cut-off Laplacian of Lax-Phillips [CV] one can deduce the following elementary bounds

$$(4.2) \quad N_{\Gamma}(\lambda) \ll \lambda^2, \quad M_{\Gamma}(\lambda) \ll \lambda^2, \quad \lambda \geq 1.$$

These bounds imply that the trace formula (4.1) holds for a larger class of functions. In particular, it can be applied to the heat kernel  $k_t(u)$ . Its spherical Fourier transform



equals  $h_t(r) = e^{-t(1/4+r^2)}$ ,  $t > 0$ . If we insert  $h_t$  into the trace formula we get the following asymptotic expansion as  $t \rightarrow 0$ .

$$(4.3) \quad \sum_j e^{-t\lambda_j} - \frac{1}{4\pi} \int_{\mathbb{R}} e^{-t(1/4+r^2)} \frac{\phi'}{\phi}(1/2 + ir) dr$$

$$= \frac{\text{Area}(\Gamma \backslash \mathbb{H})}{4\pi t} + \frac{a \log t}{\sqrt{t}} + \frac{b}{\sqrt{t}} + O(1)$$

for certain constants  $a, b \in \mathbb{R}$ . Using [Se2, (8.8), (8.9)] it follows that the winding number  $M_\Gamma(\lambda)$  is monotonic increasing for  $r \gg 0$ . Therefore we can apply a Tauberian theorem to (4.3) and we get the Weyl law (1.4).

In general, we cannot estimate separately the counting function and the winding number. For congruence subgroups, however, the entries of the scattering matrix can be expressed in terms of well-known analytic functions. For  $\Gamma(N)$  the determinant of the scattering matrix  $\phi(s)$  has been computed by Huxley [Hu]. It has the form

$$(4.4) \quad \phi(s) = (-1)^l A^{1-2s} \left( \frac{\Gamma(1-s)}{\Gamma(s)} \right)^k \prod_{\chi} \frac{L(2-2s, \bar{\chi})}{L(2s, \chi)},$$

where  $k, l \in \mathbb{Z}$ ,  $A > 0$ , the product runs over Dirichlet characters  $\chi$  to some modulus dividing  $N$  and  $L(s, \chi)$  is the Dirichlet  $L$ -function with character  $\chi$ . Especially for  $\Gamma(1)$  we have

$$(4.5) \quad \phi(s) = \sqrt{\pi} \frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)},$$

where  $\zeta(s)$  denotes the Riemann zeta function.

Using Stirling's approximation formula to estimate the logarithmic derivative of the Gamma function and standard estimations for the logarithmic derivative of Dirichlet  $L$ -functions on the line  $\text{Re}(s) = 1$  [Pr, Theorem 7.1], we get

$$(4.6) \quad \frac{\phi'}{\phi}(1/2 + ir) = O(\log(4 + |r|)), \quad |r| \rightarrow \infty.$$

This implies that

$$(4.7) \quad M_{\Gamma(N)}(\lambda) \ll \lambda \log \lambda.$$

Together with (1.4) we obtain Weyl's law for the point spectrum

$$(4.8) \quad N_{\Gamma(N)}(\lambda) \sim \frac{\text{Area}(X(N))}{4\pi} \lambda^2, \quad \lambda \rightarrow \infty,$$

which is due to Selberg [Se2, p.668]. A similar formula holds for other congruence groups such as  $\Gamma_0(N)$ . In particular, (4.8) implies that for congruence groups  $\Gamma$  there exist infinitely many linearly independent Maass cusp forms.

A proof of the Weyl law (4.8) which avoids the use of the constant terms of the Eisenstein series has recently been given by Lindenstrauss and Venkatesh [LV]. Their approach is based on the construction of convolution operators with purely cuspidal image.

Neither of these methods give good estimates of the remainder term. One approach to obtain estimates of the remainder term is based on the Selberg zeta function

$$Z_\Gamma(s) = \prod_{\{\gamma\}_\Gamma} \prod_{k=0}^{\infty} (1 - e^{-(s+k)\ell(\gamma)}), \quad \operatorname{Re}(s) > 1,$$

where the outer product runs over the primitive hyperbolic conjugacy classes in  $\Gamma$  and  $\ell(\gamma)$  is the length of the closed geodesic associated to  $\{\gamma\}_\Gamma$ . The infinite product converges absolutely in the indicated half-plane and admits an analytic continuation to the whole complex plane. If  $\lambda = 1/4 + r^2$ ,  $r \in \mathbb{R} \cup i(1/2, 1]$ , is an eigenvalue of  $\Delta$ , then  $s_0 = 1/2 + ir$  is a zero of  $Z_\Gamma(s)$ . Using this fact and standard methods of analytic number theory one can derive the following strong form of the Weyl law [Hj, Theorem 2.28], [Ve, Theorem 7.3].

**Theorem 4.1.** *Let  $m$  be the number of cusps of  $\Gamma \backslash \mathbb{H}$ . There exists  $c > 0$  such that*

$$N_\Gamma(\lambda) + M_\Gamma(\lambda) = \frac{\operatorname{Area}(\Gamma \backslash \mathbb{H})}{4\pi} \lambda^2 - \frac{m}{\pi} \lambda \log \lambda + c\lambda + O(\lambda(\log \lambda)^{-1})$$

as  $\lambda \rightarrow \infty$ .

Together with (4.7) we obtain Weyl's law with remainder term.

**Theorem 4.2.** *For every  $N \in \mathbb{N}$  we have*

$$N_{\Gamma(N)}(\lambda) = \frac{\operatorname{Area}(X(N))}{4\pi} \lambda^2 + O(\lambda \log \lambda)$$

as  $\lambda \rightarrow \infty$ .

The use of the Selberg zeta function to estimate the remainder term is limited to rank one cases. However, the remainder term can also be estimated by Hörmander's method using the trace formula as in the compact case. We describe the main steps. Let  $\varepsilon > 0$  such that  $\ell(\gamma) > \varepsilon$  for all hyperbolic conjugacy classes  $\{\gamma\}_{\Gamma(N)}$ . Choose  $g \in C_c^\infty(\mathbb{R})$  to be even and such that  $\operatorname{supp} g \subset (-\varepsilon, \varepsilon)$ . Let  $h(z) = \int_{\mathbb{R}} g(r) e^{-irz} dr$ . Then the hyperbolic contribution in the trace formula (4.1) drops out. We symmetrize the spectrum by  $r_{-j} = -r_j$ ,  $j \in \mathbb{N}$ . Then for each  $t \in \mathbb{R}$  we have

$$\begin{aligned} \sum_j h(t - r_j) &= \frac{\operatorname{Area}(X(N))}{2\pi} \int_{\mathbb{R}} h(t - r) r \tanh(\pi r) dr \\ (4.9) \quad &+ \frac{1}{2\pi} \int_{-\infty}^{\infty} h(t - r) \frac{\phi'}{\phi}(1/2 + ir) dr - \frac{1}{2} \phi(1/2) h(t) \\ &- \frac{m}{\pi} \int_{-\infty}^{\infty} h(t - r) \frac{\Gamma'}{\Gamma}(1 + ir) dr + \frac{m}{2} h(t) - 2m \ln 2 g(0). \end{aligned}$$

Now we need to estimate the behavior of the terms on the right hand side as  $|t| \rightarrow \infty$ . The first integral has been already considered in (2.7). It is of order  $O(|t|)$ . To deal with the second integral we use (4.6). This implies

$$(4.10) \quad \int_{\mathbb{R}} h(t-r) \frac{\phi'}{\phi} (1/2 + ir) dr = O(\log(|t|)), \quad |t| \rightarrow \infty.$$

Using Stirling's formula we get

$$\int_{\mathbb{R}} h(t-r) \frac{\Gamma'}{\Gamma} (1 + ir) dr = O(\log(|t|)), \quad |t| \rightarrow \infty.$$

The remaining terms are bounded. Combining these estimations, we get

$$\sum_j h(t-r_j) = O(|t|), \quad |t| \rightarrow \infty.$$

Therefore, Lemma 2.2 holds also in the present case. It remains to establish the analog of Lemma 2.4. Using (2.10) and (4.6) we get

$$(4.11) \quad \int_{-\lambda}^{\lambda} \int_{\mathbb{R}} h(t-r) \frac{\phi'}{\phi} (1/2 + it) dr dt = O(\lambda \log \lambda).$$

Similarly, using Stirling's formula and (2.10), we obtain

$$(4.12) \quad \int_{-\lambda}^{\lambda} \int_{\mathbb{R}} h(t-r) \frac{\Gamma'}{\Gamma} (1 + it) dr dt = O(\lambda \log \lambda).$$

The integral of the remaining terms is of order  $O(\lambda)$ . Thus we obtain

$$(4.13) \quad \int_{-\lambda}^{\lambda} \sum_{j=-\infty}^{\infty} h(t-r_j) dt = \frac{\text{Area}(X(N))}{2\pi} \lambda^2 + O(\lambda \log \lambda)$$

as  $\lambda \rightarrow \infty$ . Now we proceed in exactly the same way as in the compact case. Using Lemma 2.3 and (4.13), Theorem 4.2 follows.  $\square$

The Weyl law shows that for congruence groups Maass cusp forms exist in abundance. In general very little is known. Let  $\Gamma$  be any discrete, co-finite subgroup of  $\text{SL}(2, \mathbb{R})$ . Then by Donnelly [Do] the following general bound is known

$$\limsup_{\lambda \rightarrow \infty} \frac{N_{\Gamma}^{\text{cus}}(\lambda)}{\lambda} \leq \frac{\text{Area}(\Gamma \backslash \mathbb{H})}{4\pi}.$$

A group  $\Gamma$  for which the equality is attained is called *essentially cuspidal* by Sarnak [Sa2]. By (4.8),  $\Gamma(N)$  is essentially cuspidal. The study of the behavior of eigenvalues under deformations of  $\Gamma$ , initiated Phillips and Sarnak [PS1], [PS2], supports the conjecture that essential cuspidality may be limited to special arithmetic groups.

The consideration of the behavior of cuspidal eigenvalues under deformations was started by Colin de Verdière [CV] in the more general context of metric perturbations. One of his main results [CV, Théorème 7] states that under a generic compactly supported conformal perturbation of the hyperbolic metric of  $\Gamma \backslash \mathbb{H}$  all Maass cusp forms are dissolved. This

means that each point  $s_j = 1/2 + ir_j$ ,  $r_j \in \mathbb{R}$ , such that  $\lambda_j = s_j(1 - s_j)$  is an eigenvalue moves under the perturbation into the half-plane  $\operatorname{Re}(s) < 1/2$  and becomes a pole of the scattering matrix  $C(s)$ .

In the present context we are only interested in deformations such that the curvature stays constant. Such deformations are given by curves in the Teichmüller space  $\mathcal{T}(\Gamma)$  of  $\Gamma$ . The space  $\mathcal{T}(\Gamma)$  is known to be finite-dimensional and therefore, it is by no means clear that the results of [CV] will continue to hold for perturbations of this restricted type. For  $\Gamma(N)$  this problem has been studied in [PS1], [PS2]. One of the main results is an analog of Fermi's golden rule which gives a sufficient condition for a cusp form of  $\Gamma(N)$  to be dissolved under a deformation in  $\mathcal{T}(\Gamma(N))$ . Based on these results, Sarnak made the following conjecture [Sa2]:

**Conjecture.**

- (a) *The generic  $\Gamma$  in a given Teichmüller space of finite area hyperbolic surfaces is not essentially cuspidal.*
- (b) *Except for the Teichmüller space of the once punctured torus, the generic  $\Gamma$  has only finitely many eigenvalues.*

## 5. HIGHER RANK

In this section we consider an arbitrary locally symmetric space  $\Gamma \backslash S$  defined by an arithmetic subgroup  $\Gamma \subset \mathbf{G}(\mathbb{Q})$ , where  $\mathbf{G}$  is a semi-simple algebraic group over  $\mathbb{Q}$  with finite center,  $G = \mathbf{G}(\mathbb{R})$  and  $S = G/K$ . The basic example will be  $\mathbf{G} = \operatorname{SL}(n)$  and  $\Gamma = \Gamma(N)$ , the principal congruence subgroup of  $\operatorname{SL}(n, \mathbb{Z})$  of level  $N$  which consists of all  $\gamma \in \operatorname{SL}(n, \mathbb{Z})$  such that  $\gamma \equiv \operatorname{Id} \pmod{N}$ .

Let  $\Delta$  be the Laplacian of  $\Gamma \backslash S$ , and let  $\bar{\Delta}$  be the closure of  $\Delta$  in  $L^2$ . Then  $\bar{\Delta}$  is a non-negative self-adjoint operator in  $L^2(\Gamma \backslash S)$ . The properties of its spectral resolution can be derived from the known structure of the spectral resolution of the regular representation  $R_\Gamma$  of  $G$  on  $L^2(\Gamma \backslash G)$  [La1], [BG]. In this way we get the following generalization of Proposition 3.1.

**Proposition 5.1.** *The spectrum of  $\bar{\Delta}$  is the union of a point spectrum  $\sigma_{pp}(\bar{\Delta})$  and an absolutely continuous spectrum  $\sigma_{ac}(\bar{\Delta})$ .*

- 1) *The point spectrum consists of eigenvalues  $0 = \lambda_0 < \lambda_1 \leq \dots$  of finite multiplicities with no finite point of accumulation.*
- 2) *The absolutely continuous spectrum equals  $[b, \infty)$  for some  $b > 0$ .*

The theory of Eisenstein series [La1] provides a complete set of generalized eigenfunctions for  $\Delta$ . The corresponding wave packets span the absolutely continuous subspace  $L_c^2(\Gamma \backslash S)$ . This allows us to determine the constant  $b$  explicitly in terms of the root structure. The statement about the point spectrum was proved in [BG, Theorem 5.5].

Let  $L_{\text{dis}}^2(\Gamma \backslash S)$  be the closure of the span of all eigenfunctions. It contains the subspace of cusp forms  $L_{\text{cus}}^2(\Gamma \backslash S)$ . We recall its definition. Let  $\mathbf{P} \subset \mathbf{G}$  be a parabolic subgroup

defined over  $\mathbb{Q}$  [Bo]. Let  $P = \mathbf{P}(\mathbb{R})$ . This is a cuspidal parabolic subgroup of  $G$  and all cuspidal parabolic subgroups arise in this way. Let  $\mathbf{N}_P$  be the unipotent radical of  $\mathbf{P}$  and let  $N_P = \mathbf{N}_P(\mathbb{R})$ . Then  $N_P \cap \Gamma \backslash N_P$  is compact. A cusp form is a smooth function  $\phi$  on  $\Gamma \backslash S$  which is a joint eigenfunction of the ring  $\mathcal{D}(S)$  of invariant differential operators on  $S$ , and which satisfies

$$(5.1) \quad \int_{N_P \cap \Gamma \backslash N_P} \phi(nx) \, dn = 0$$

for all cuspidal parabolic subgroups  $P \neq G$ . Each cusp form is rapidly decreasing and hence square integrable. Let  $L_{\text{cus}}^2(\Gamma \backslash S)$  be the closure in  $L^2(\Gamma \backslash S)$  of the linear span of all cusp forms. Then  $L_{\text{cus}}^2(\Gamma \backslash S)$  is an invariant subspace of  $\bar{\Delta}$  which is contained in  $L_{\text{dis}}^2(\Gamma \backslash S)$ .

Let  $L_{\text{res}}^2(\Gamma \backslash S)$  be the orthogonal complement of  $L_{\text{cus}}^2(\Gamma \backslash S)$  in  $L_{\text{dis}}^2(\Gamma \backslash S)$ , i.e., we have an orthogonal decomposition

$$L_{\text{dis}}^2(\Gamma \backslash S) = L_{\text{cus}}^2(\Gamma \backslash S) \oplus L_{\text{res}}^2(\Gamma \backslash S).$$

It follows from Langlands's theory of Eisenstein systems that  $L_{\text{res}}^2(\Gamma \backslash S)$  is spanned by *iterated residues* of cuspidal Eisenstein series [La1, Chapter 7]. Therefore  $L_{\text{res}}^2(\Gamma \backslash S)$  is called the *residual subspace*.

Let  $N_{\Gamma}^{\text{dis}}(\lambda)$ ,  $N_{\Gamma}^{\text{cus}}(\lambda)$ , and  $N_{\Gamma}^{\text{res}}(\lambda)$  be the counting function of the eigenvalues with eigenfunctions belonging to the corresponding subspaces. The following general results about the growth of the counting functions are known for any lattice  $\Gamma$  in a real semi-simple Lie group. Let  $n = \dim S$ . Donnelly [Do] has proved the following bound for the cuspidal spectrum

$$(5.2) \quad \limsup_{\lambda \rightarrow \infty} \frac{N_{\Gamma}^{\text{cus}}(\lambda)}{\lambda^n} \leq \frac{\text{vol}(\Gamma \backslash S)}{(4\pi)^{n/2} \Gamma\left(\frac{n}{2} + 1\right)},$$

where  $\Gamma(s)$  denotes the Gamma function. For the full discrete spectrum, we have at least an upper bound for the growth of the counting function. The main result of [Mu2] states that

$$(5.3) \quad N_{\Gamma}^{\text{dis}}(\lambda) \ll (1 + \lambda^{4n}).$$

This result implies that invariant integral operators are trace class on the discrete subspace which is the starting point for the trace formula. The proof of (5.3) relies on the description of the residual subspace in terms of iterated residues of Eisenstein series. One actually expects that the growth of the residual spectrum is of lower order than the cuspidal spectrum. For  $\text{SL}(n)$  the residual spectrum has been determined by Moeglin and Waldspurger [MW]. Combined with (5.2) it follows that for  $G = \text{SL}(n)$  we have

$$(5.4) \quad N_{\Gamma(N)}^{\text{res}}(\lambda) \ll \lambda^{d-1},$$

where  $d = \dim \text{SL}(n, \mathbb{R}) / \text{SO}(n)$ .

In [Sa2] Sarnak conjectured that if  $\text{rank}(G/K) > 1$ , each irreducible lattice  $\Gamma$  in  $G$  is essentially cuspidal in the sense that Weyl's law holds for  $N_{\Gamma}^{\text{cus}}(\lambda)$ , i.e., equality holds in (5.2). This conjecture has now been established in quite generality. A. Reznikov proved it

for congruence groups in a group  $G$  of real rank one, S. Miller [Mi] proved it for  $\mathbf{G} = \mathrm{SL}(3)$  and  $\Gamma = \mathrm{SL}(3, \mathbb{Z})$ , the author [Mu3] established it for  $\mathbf{G} = \mathrm{SL}(n)$  and a congruence group  $\Gamma$ . The method of [Mu3] is an extension of the heat equation method described in the previous section for the case of the upper half-plane. More recently, Lindenstrauss and Venkatesh [LV] proved the following result.

**Theorem 5.2.** *Let  $\mathbf{G}$  be a split adjoint semi-simple group over  $\mathbb{Q}$  and let  $\Gamma \subset \mathbf{G}(\mathbb{Q})$  be a congruence subgroup. Let  $n = \dim S$ . Then*

$$N_{\Gamma}^{\mathrm{cus}}(\lambda) \sim \frac{\mathrm{vol}(\Gamma \backslash S)}{(4\pi)^{n/2} \Gamma\left(\frac{n}{2} + 1\right)} \lambda^n, \quad \lambda \rightarrow \infty.$$

The method is based on the construction of convolution operators with pure cuspidal image. It avoids the delicate estimates of the contributions of the Eisenstein series to the trace formula. This proves existence of many cusp forms for these groups.

The next problem is to estimate the remainder term. For  $\mathbf{G} = \mathrm{SL}(n)$ , this problem has been studied by E. Lapid and the author in [LM]. Actually, we consider not only the cuspidal spectrum of the Laplacian, but the cuspidal spectrum of the whole algebra of invariant differential operators.

As  $\mathcal{D}(S)$  preserves the space of cusp forms, we can proceed as in the compact case and decompose  $L_{\mathrm{cus}}^2(\Gamma \backslash S)$  into joint eigenspaces of  $\mathcal{D}(S)$ . Given  $\lambda \in \mathfrak{a}_{\mathbb{C}}^*/W$ , let

$$\mathcal{E}_{\mathrm{cus}}(\lambda) = \{\varphi \in L_{\mathrm{cus}}^2(\Gamma \backslash S) : D\varphi = \chi_{\lambda}(D)\varphi, D \in \mathcal{D}(S)\}$$

be the associated eigenspace. Each eigenspace is finite-dimensional. Let  $m(\lambda) = \dim \mathcal{E}_{\mathrm{cus}}(\lambda)$ . Define the cuspidal spectrum  $\Lambda_{\mathrm{cus}}(\Gamma)$  to be

$$\Lambda_{\mathrm{cus}}(\Gamma) = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^*/W : m(\lambda) > 0\}.$$

Then we have an orthogonal direct sum decomposition

$$L_{\mathrm{cus}}^2(\Gamma \backslash S) = \bigoplus_{\lambda \in \Lambda_{\mathrm{cus}}(\Gamma)} \mathcal{E}_{\mathrm{cus}}(\lambda).$$

Let the notation be as in (2.18) and (2.19). Then in [LM] we established the following extension of main results of [DKV] to congruence quotients of  $S = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ .

**Theorem 5.3.** *Let  $d = \dim S$ . Let  $\Omega \subset \mathfrak{a}^*$  be a bounded domain with piecewise smooth boundary. Then for  $N \geq 3$  we have*

$$(5.5) \quad \sum_{\lambda \in \Lambda_{\mathrm{cus}}(\Gamma(N)), \lambda \in i t \Omega} m(\lambda) = \frac{\mathrm{vol}(\Gamma(N) \backslash S)}{|W|} \int_{it\Omega} \beta(i\lambda) d\lambda + O\left(t^{d-1} (\log t)^{\max(n, 3)}\right),$$

as  $t \rightarrow \infty$ , and

$$(5.6) \quad \sum_{\substack{\lambda \in \Lambda_{\mathrm{cus}}(\Gamma(N)) \\ \lambda \in B_t(0) \setminus i\mathfrak{a}^*}} m(\lambda) = O\left(t^{d-2}\right), \quad t \rightarrow \infty.$$

If we apply (5.5) and (5.6) to the unit ball in  $\mathfrak{a}^*$ , we get the following corollary.

**Corollary 5.4.** *Let  $\mathbf{G} = \mathrm{SL}(n)$  and let  $\Gamma(N)$  be the principal congruence subgroup of  $\mathrm{SL}(n, \mathbb{Z})$  of level  $N$ . Let  $S = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$  and  $d = \dim S$ . Then for  $N \geq 3$  we have*

$$N_{\Gamma(N)}^{\mathrm{cus}}(\lambda) = \frac{\mathrm{vol}(\Gamma(N) \backslash S)}{(4\pi)^{d/2} \Gamma\left(\frac{d}{2} + 1\right)} \lambda^d + O\left(\lambda^{d-1} (\log \lambda)^{\max(n, 3)}\right), \quad \lambda \rightarrow \infty.$$

The condition  $N \geq 3$  is imposed for technical reasons. It guarantees that the principal congruence subgroup  $\Gamma(N)$  is neat in the sense of Borel, and in particular, has no torsion. This simplifies the analysis by eliminating the contributions of the non-unipotent conjugacy classes in the trace formula.

Note that  $\Lambda_{\mathrm{cus}}(\Gamma(N)) \cap i\mathfrak{a}^*$  is the cuspidal tempered spherical spectrum. The Ramanujan conjecture [Sa3] for  $\mathrm{GL}(n)$  at the Archimedean place states that

$$\Lambda_{\mathrm{cus}}(\Gamma(N)) \subset i\mathfrak{a}^*$$

so that (5.6) is empty, if the Ramanujan conjecture is true. However, the Ramanujan conjecture is far from being proved. Moreover, it is known to be false for other groups  $\mathbf{G}$  and (5.6) is what one can expect in general.

The method to prove Theorem 5.3 is an extension of the method of [DKV]. The Selberg trace formula, which is one of the basic tools in [DKV], needs to be replaced by the Arthur trace formula [A1], [A2]. This requires to change the framework and to work with the adelic setting. It is also convenient to replace  $\mathrm{SL}(n)$  by  $\mathrm{GL}(n)$ .

Again, one of the main issues is to estimate the terms in the trace formula which are associated to Eisenstein series. Roughly speaking, these terms are a sum of integrals which generalize the integral

$$\int_{-\infty}^{\infty} h(r) \frac{\phi'}{\phi}(1/2 + ir) dr$$

in (4.1). The sum is running over Levi components of parabolic subgroups and square integrable automorphic forms on a given Levi component. The functions which generalize  $\phi(s)$  are obtained from the constant terms of Eisenstein series. In general, they are difficult to describe. The main ingredients are logarithmic derivatives of automorphic  $L$ -functions associated to automorphic forms on the Levi components. As example consider  $\mathbf{G} = \mathrm{SL}(3)$ ,  $\Gamma = \mathrm{SL}(3, \mathbb{Z})$ , and a standard maximal parabolic subgroup  $P$  which has the form

$$P = \left\{ \begin{pmatrix} m_1 & X \\ 0 & m_2 \end{pmatrix} \mid m_i \in \mathrm{GL}(n_i, \mathbb{R}), \det m_1 \cdot \det m_2 = 1 \right\},$$

with  $n_1 + n_2 = 3$ . Thus there are exactly two standard maximal parabolic subgroups. The standard Levi component of  $P$  is

$$L = \left\{ \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \mid m_i \in \mathrm{GL}(n_i, \mathbb{R}), \det m_1 \cdot \det m_2 = 1 \right\}.$$

So  $L$  is isomorphic to  $\mathrm{GL}(2, \mathbb{R})$ . The Eisenstein series are associated to Maass cusp forms on  $\Gamma(1) \backslash \mathbb{H}$ . The constant terms of the Eisenstein series are described in [Go, Proposition

10.11.2]. Let  $f$  be a Maass cusp form for  $\Gamma(1)$  and let  $\Lambda(s, f)$  be the completed  $L$ -function of  $f$  defined by (3.4). Then the relevant constant term of the Eisenstein series associated to  $f$  is given by

$$\frac{\Lambda(s, f)}{\Lambda(1 + s, f)}.$$

To proceed one needs a bound similar to (4.6). Assume that  $\Delta f = (1/4 + r^2)f$ . Using the analytic properties of  $\Lambda(s, f)$  one can show that for  $T \geq 1$

$$(5.7) \quad \int_{-T}^T \frac{\Lambda'}{\Lambda}(1 + it, f) dt \ll T \log(T + |r|).$$

This is the key result that is needed to deal with the contribution of the Eisenstein series to the trace formula.

The example demonstrates a general feature of spectral theory on locally symmetric spaces. Harmonic analysis on higher rank spaces requires the knowledge of the analytic properties of automorphic  $L$ -functions attached to cusp forms on lower rank groups. For  $\mathrm{GL}(n)$ , the corresponding  $L$ -functions are Rankin-Selberg convolutions  $L(s, \phi_1 \times \phi_2)$  of automorphic cusp forms on  $\mathrm{GL}(n_i)$ ,  $i = 1, 2$ , where  $n_1 + n_2 = n$  (cf. [Bu], [Go] for their definition). The analytic properties of these  $L$ -functions are well understood so that estimates similar to (5.7) can be established. For other groups  $\mathbf{G}$  (except for some low dimensional cases) our current knowledge of the analytic properties of the corresponding  $L$ -functions is not sufficient to prove estimates like (5.7). Only partial results exist [CPS]. This is one of the main obstacles to extend Theorem 5.3 to other groups.

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